The crossing number of folded hypercubes *

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Abstract

The *crossing number* of a graph G is the minimum number of pairwise intersections of edges in a drawing of G. The *n-dimensional folded hypercube* FQ_n is a graph obtained from n-dimensional hypercube by adding all complementary edges. In this paper, we obtain upper and lower bounds of the crossing number of FQ_n.

Keywords: *Drawing*; *Crossing number*; *Folded hypercube*

1 Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The *crossing number* $cr(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane. In the past thirty years, it turned out that crossing number played an important role not only in various fields of discrete and computational geometry (see [2, 12, 14, 16]), but also in VLSI theory and wiring layout problems (see [1, 9, 10, 13]). For this reason, the study of crossing number of some popular parallel network topologies such as hypercube and its variants which have good topological properties and applications in VLSI theory, would be of theoretical importance and practical value. An n-dimensional hypercube Q_n is a graph in which the nodes can be one-to-one labeled

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with 0-1 binary sequences of length n, so that the labels of any two adjacent nodes differ in exactly one bit. Determining the crossing number of an arbitrary graph is proved to be NP-complete [7]. Even for hypercube, for a long time the only known result on the exact value of crossing number of Q_n has been $cr(Q_3) = 0$, $cr(Q_4) = 8$ [3], $cr(Q_5) \leq 56$ [11]. Hence, it is more practical to find upper and lower bounds of crossing numbers of hypercube and its variants. Concerned with upper bound of crossing number of hypercube, Erdős and Guy [5] in 1973 conjectured the following:

$$cr(Q_n) \leq \frac{5}{32}4^n - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2}.$$

In 2008, Sykora and Vrt'o [6] constructed a drawing of Q_n in the plane which has the conjectured number of crossings mentioned above. Early in 1993 they [15] also proved a lower bound of $cr(Q_n)$:

$$cr(Q_n) > \frac{1}{20}4^n - (n^2+1)2^{n-1}.$$

Since the hypercube does not have the smallest possible diameter for its resources, to achieve smaller diameter with the same number of nodes and links as an n-dimensional hypercube, a variety of hypercube variants were proposed. Folded hypercube is one of these variants. The *n-dimensional folded hypercube FQ_n* was proposed by El-Amawy and Latifi [4] in 1991. The folded hypercube has many appealing features of the n-dimensional hypercube such as node and edge symmetry. In addition, it has been shown to be superior over the n-dimensional hypercube in many communication aspects such as halved diameter, better average distance, shorter delay in communication links, less message traffic density and lower cost. Therefore, it would be more attractive to study the crossing number of the folded hypercube.

The *n-dimensional folded hypercube FQ_n* is a graph obtained from Q_n by adding all complementary edges, which join a vertex $x = x_1x_2 \ldots x_n$ to another vertex $\overline{x} = \overline{x}_1\overline{x}_2 \ldots \overline{x}_n$ for every $x \in V(Q_n)$, where $\overline{x}_i = \{0,1\} \setminus \{x_i\}$. The graphs shown in Figure 1.1 are FQ_1, FQ_2, FQ_3 and FQ_4, respectively, where thick lines represent the complementary edges. It is easy to see that $FQ_1 \cong K_2$, $FQ_2 \cong K_4$ and $FQ_3 \cong K_{4,4}$ (see Figure 1.2). So $cr(FQ_1) = cr(FQ_2) = 0$ and $cr(FQ_3) = 4$.

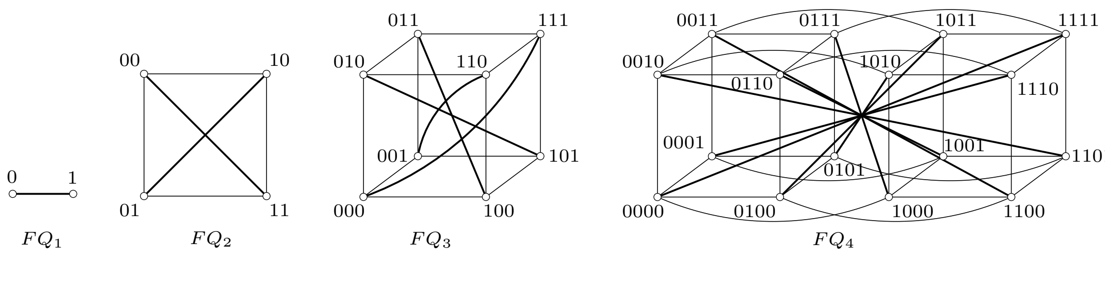

Figure 1.1: Folded hypercube FQ_1, FQ_2, FQ_3 and FQ_4

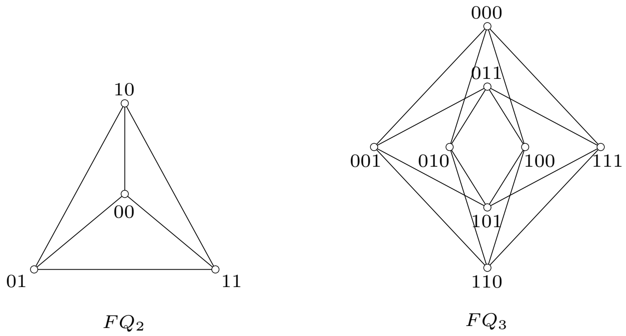

Figure 1.2: Drawings of $FQ_2 \cong K_4$ and $FQ_3 \cong K_{4,4}$

In this paper, we prove the following bounds of $cr(FQ_n)$:

$$\frac{4^n}{20 \times (1 - \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\lceil \frac{n}{2} \rceil + 1}})^2} - (n^2 + 2n + 4)2^{n-1} < cr(FQ_n) \leq \frac{11}{32}4^n - (n^2 + 3n)2^{n-3}.$$

2 Upper bound

A drawing of G is said to be a *good* drawing, provided that no edge crosses itself, no adjacent edges cross each other, no two edges cross more than once, and no three edges cross in a point. It is well known that the crossing number of a graph is attained only in *good* drawings of the graph. So, we always assume that all drawings throughout this paper are good drawings. For a good drawing D of a graph G, let $\nu_D(G)$ be the number of crossings in D.

For a binary string $x_1x_2 \cdots x_n$, let

$$\mathscr{D}(x_1x_2 \cdots x_n) = 2^{n-1}x_1 + 2^{n-2}x_2 + \cdots + 2^0 x_n$$

be the corresponding decimal number of $x_1x_2 \cdots x_n$. For any integers $n > m \geq 1$ and binary string $x_1x_2 \cdots x_m$, let

$$\mathcal{F}^n_{x_1 \cdots x_m} = \{y_1y_2 \cdots y_n : y_1, y_2, \ldots, y_n \in \{0,1\}, y_i = x_i \text{ for all } i \in \{1, 2, \ldots, m\}\}.$$

For a vertex subset S of a graph G, let $\langle S \rangle$ be the subgraph of G induced by S.

We need the following observations which will be useful for the proofs of the upper bound.

Observation 2.1. *Let $\mathcal{F}^n_{x_1 \cdots x_m}$ be a vertex subset of FQ_n, where $n > m \geq 1$ and $x_1, \ldots, x_m \in \{0,1\}$. Then there exists an isomorphism μ of $\langle \mathcal{F}^n_{x_1 \cdots x_m} \rangle$ onto Q_{n-m} such that $\mu(x_1 \cdots x_m x_{m+1} \cdots x_n) = x_{m+1} \cdots x_n$ for all $x_{m+1}, \ldots, x_n \in \{0,1\}$.*

Observation 2.2. *For any $m \geq 1$, let R and S be two non-horizontal bunches of m parallel lines starting from points $(0,0), (1,0), \ldots, (m-1,0)$ respectively (see Figure 2.1), which are above the real axis x. Then the number of crossings between R and S is $\frac{m(m-1)}{2}$.*

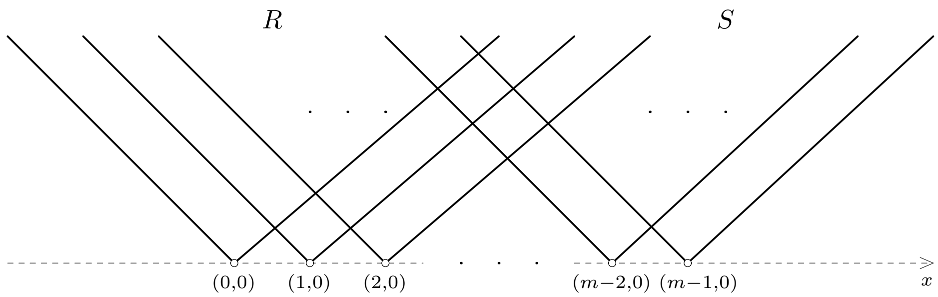

Figure 2.1: The crossings between two bunches of m parallel lines R and S

Now we shall introduce a recursive drawing Γ_n of Q_n. Consider the real axis x in the 2-dimensional Euclidean plane. Let Γ_{n-1} be a drawing of Q_{n-1} in the plane such that all vertices of Q_{n-1} are drawn in the points $(0,0),(1,0),\ldots,(2^{n-1}-1,0)$. Produce an identical drawing to Γ_{n-1} in the points $(2^{n-1},0),(2^{n-1}+1,0),\ldots,(2^n-1,0)$. Then join point $(i,0)$ and point $(2^{n-1}+i,0)$ for all $i\in\{0,1,\ldots,2^{n-1}-1\}$ by "parallel" curves. In Figure 2.2, we show as an example drawings of Γ_1, Γ_2 and Γ_3, where the "parallel" curves joining $(i,0)$ and $(2^{n-1}+i,0)$ are drawn in red.

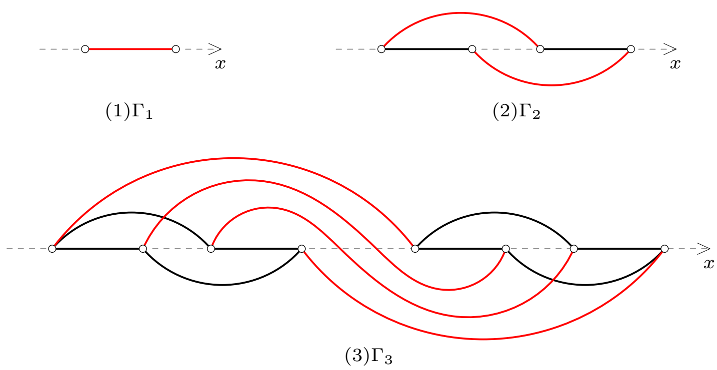

Figure 2.2: Drawings of Γ_1, Γ_2 and Γ_3

By the construction of Γ_n, it is easy to observe

Observation 2.3. *In the drawing Γ_n, any vertex $x_1x_2\cdots x_n\in V(Q_n)$ and its complementary vertex $\bar{x}_1\bar{x}_2\cdots\bar{x}_n$ are drawn in point $(\mathscr{D}(x_1x_2\cdots x_n),0)$ and point $((2^n-1)-\mathscr{D}(x_1x_2\cdots x_n),0)$, respectively.*

We still need to introduce a necessary definition.

Definition 2.1. *In the drawing Γ_n, for a vertex v of Q_n, let $C_a^n(v)$ and $C_b^n(v)$ be the number of curves which cover v from the upper semi-plane of real axis x and from the lower semi-plane of real axis x, respectively (see Figure 2.3).*

In Figure 2.3, the curve joining u_1 and u_2 covers v_1,v_2 from the upper semi-plane of real axis x, and covers v_3 from the lower semi-plane of real axis x.

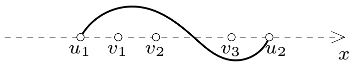

Figure 2.3: Vertices covered by a curve

Lemma 2.1. *For* $n \geq 1$,

$$\sum_{v \in V(Q_n)} C_a^n(v) = \sum_{v \in V(Q_n)} C_b^n(v) = 4^{n-1} - (n+1)2^{n-2} \ .$$

Proof. We argue by induction on n. Lemma 2.1 is true for $n = 1$. Now assume $n > 1$. By the construction of Γ_n, we see that the curves joining point $(i, 0)$ and point $(2^{n-1} + i, 0)$ would cover points $(i+1, 0), (i+2, 0), \ldots, (2^{n-1} - 1, 0)$ from the upper semi-plane of axis x and cover points $(2^{n-1}, 0), (2^{n-1} + 1, 0), \ldots, (2^{n-1} + i - 1, 0)$ from the lower semi-plane of axis x for all $i \in \{0, 1, \ldots, 2^{n-1} - 1\}$. Therefore, it follows that

$$\sum_{v \in V(Q_n)} C_a^n(v) = (1 + 2 + \cdots + (2^{n-1} - 1)) + 2 \times \sum_{v \in V(Q_{n-1})} C_a^{n-1}(v)$$

and

$$\sum_{v \in V(Q_n)} C_b^n(v) = (1 + 2 + \cdots + (2^{n-1} - 1)) + 2 \times \sum_{v \in V(Q_{n-1})} C_b^{n-1}(v).$$

Then the lemma follows from immediate verification. $\square$

Lemma 2.2. *For* $n \geq 1$,

$$\nu_{\Gamma_n}(Q_n) = 4^{n-1} - (n^2 + n + 2)2^{n-3} \ .$$

Proof. The proof will be by induction on n. Lemma 2.2 is true for $n = 1$ since $\nu_{\Gamma_1}(Q_1) = 0$. Now assume $n > 1$. We see that the curves joining point $(i, 0)$ and point $(2^{n-1} + i, 0)$ would cross all curves which cover point $(i, 0)$ from the upper semi-plane of axis x and belong to the induced subgraph $\langle \mathcal{F}_0^n \rangle$ of Q_n, similarly would cross all curves which cover point $(2^{n-1} + i, 0)$ from the lower semi-plane of axis x and belong to the induced subgraph $\langle \mathcal{F}_1^n \rangle$ for all $i \in \{0, 1, \ldots, 2^{n-1} - 1\}$. Therefore, it follows from Lemma 2.1 that

$$\begin{aligned}
\nu_{\Gamma_n}(Q_n) &= 2 \times \nu_{\Gamma_{n-1}}(Q_{n-1}) + \sum_{v \in V(Q_{n-1})} C_a^{n-1}(v) + \sum_{v \in V(Q_{n-1})} C_b^{n-1}(v) \\
&= 2 \times \nu_{\Gamma_{n-1}}(Q_{n-1}) + (4^{n-2} - n \times 2^{n-3}) \times 2 \\
&= 2 \times \nu_{\Gamma_{n-1}}(Q_{n-1}) + 2^{2n-3} - n \times 2^{n-2} \\
&= 2^2 \times \nu_{\Gamma_{n-2}}(Q_{n-2}) + 2^{2n-4} - (n-1) \times 2^{n-2} + 2^{2n-3} - n \times 2^{n-2} \\
&\vdots \\
&= 2^{n-1} \times \nu_{\Gamma_1}(Q_1) + (2^{n-1} + 2^n + \cdots + 2^{2n-3}) - (2 + 3 + \cdots + n) \times 2^{n-2} \\
&= 2^{n-1} \times (2^{n-1} - 1) + (n^2 + n - 2) \times 2^{n-3} \\
&= 4^{n-1} - (n^2 + n + 2)2^{n-3}.
\end{aligned}$$

$\square$

Theorem 2.1. *For $n \geq 3$,*

$$cr(FQ_n) \leq \frac{11}{32}4^n - (n^2 + 3n)2^{n-3}.$$

Proof. To prove the theorem, it suffices to construct a drawing D_n of FQ_n with $\nu_{D_n}(FQ_n) = \frac{11}{32}4^n - (n^2 + 3n)2^{n-3}$ for all $n \geq 3$. If $n = 3$, let D_3 be the drawing shown in Figure 2.4, in which the number of crossings is $4 = \frac{11}{32} \times 4^3 - (3^2 + 3 \times 3) \times 2^{3-3}$.

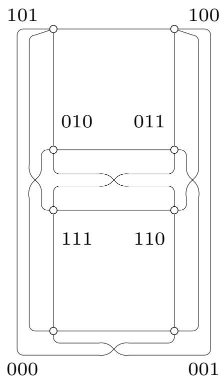

Figure 2.4: A drawing D_3 of FQ_3 with 4 crossings

Now assume $n > 3$. We set eight positive directions shown in Figure 2.5, where the red arrows stand for the positive directions. By Observation 2.1, in the drawing D_3, we replace every vertex $x_1x_2x_3 \in V(FQ_3)$ in the "small" neighborhood of $x_1x_2x_3$ by the induced subgraph $\langle \mathcal{F}^n_{x_1x_2x_3} \rangle$ of FQ_n, in which the drawing of $\langle \mathcal{F}^n_{x_1x_2x_3} \rangle$ is coincident with Γ_{n-3} and the positive direction of Γ_{n-3} in Figure 2.2 is identical to that of Figure 2.5.

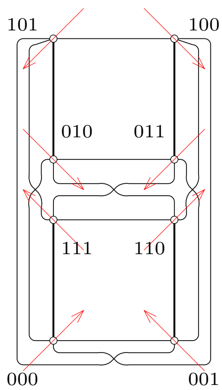

Figure 2.5: D_3 with positive direction(Thick lines represent the complementary edges)

Then every edge e incident with vertex $x_1x_2x_3$ and vertex $y_1y_2y_3$ in D_3 will be replaced by a bunch of 2^{n-3} edges which are incident with $\mathcal{F}^n_{x_1x_2x_3}$ and $\mathcal{F}^n_{y_1y_2y_3}$ and drawn along the original edge e in D_3. Combined with Observation 2.3 and the arrangement of positive directions shown in Figure 2.5, we conclude that all edges in an arbitrary bunch will be parallel. Therefore, the total number of crossings in D_n will be the number of crossings in

the small neighborhoods of all induced subgraphs $\langle \mathcal{F}^n_{x_1x_2x_3} \rangle$ plus $2^{n-3} \cdot 2^{n-3} \cdot \nu_{D_3}(FQ_3)$, where $x_1x_2x_3 \in V(FQ_3)$.

For the conveniences of the reader, we offer drawings for FQ_4, FQ_5, FQ_6 and FQ_7 in Figures 2.6-2.9 obtained according to the rules of construction mentioned above, where the vertices are represented by decimal numbers, the edges of $\langle \mathcal{F}^n_{x_1x_2x_3} \rangle$ are drawn in red and the rest edges are drawn in blue.

Claim A. For any vertex $x_1x_2x_3$ of FQ_3, the number of crossings in the small neighborhood of the induced subgraph $\langle \mathcal{F}^n_{x_1x_2x_3} \rangle$ in the drawing D_n is

$$9 \cdot 4^{n-4} - (n^2 + 3n)2^{n-6}.$$

Proof of Claim A. Let $E_r = E(\langle \mathcal{F}^n_{x_1x_2x_3} \rangle)$ and E_b be the set consisting of all edges of FQ_n which have exactly one end in $V(\langle \mathcal{F}^n_{x_1x_2x_3} \rangle)$. Then the number of crossings in the small neighborhood of $\langle \mathcal{F}^n_{x_1x_2x_3} \rangle$ is

$$\nu_{D_n}(E_r) + \nu_{D_n}(E_b) + \nu_{D_n}(E_r, E_b).$$

By Observation 2.1 and Lemma 2.2, we have that

$$\nu_{D_n}(E_r) = 4^{n-4} - ((n-3)^2 + (n-3) + 2)2^{n-6}.$$

By Observation 2.2, we have that

$$\nu_{D_n}(E_b) = 2 \cdot \frac{2^{n-3}(2^{n-3} - 1)}{2}.$$

By Lemma 2.1, we have that

$$\nu_{D_n}(E_r, E_b) = 4 \cdot (4^{n-4} - (n-2)2^{n-5}).$$

Then Claim A holds by immediate verification. □

By Claim A, we have that $\nu_{D_n}(FQ_n) = 8 \cdot (9 \cdot 4^{n-4} - (n^2 + 3n)2^{n-6}) + 2^{n-3} \cdot 2^{n-3} \cdot \nu_{D_3}(FQ_3) = \frac{11}{32}4^n - (n^2 + 3n)2^{n-3}$. This completes the proof of the theorem. □

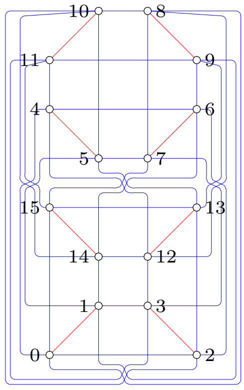

Figure 2.6: A drawing D_4 of FQ_4

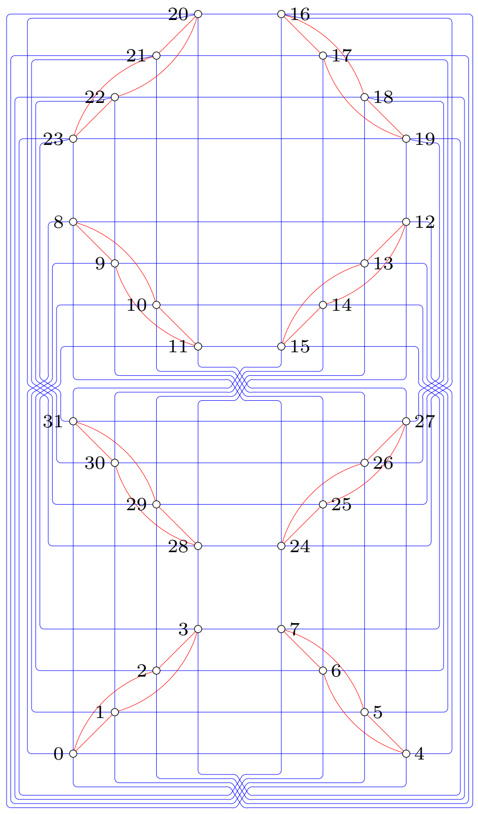

Figure 2.7: A drawing D_5 of FQ_5

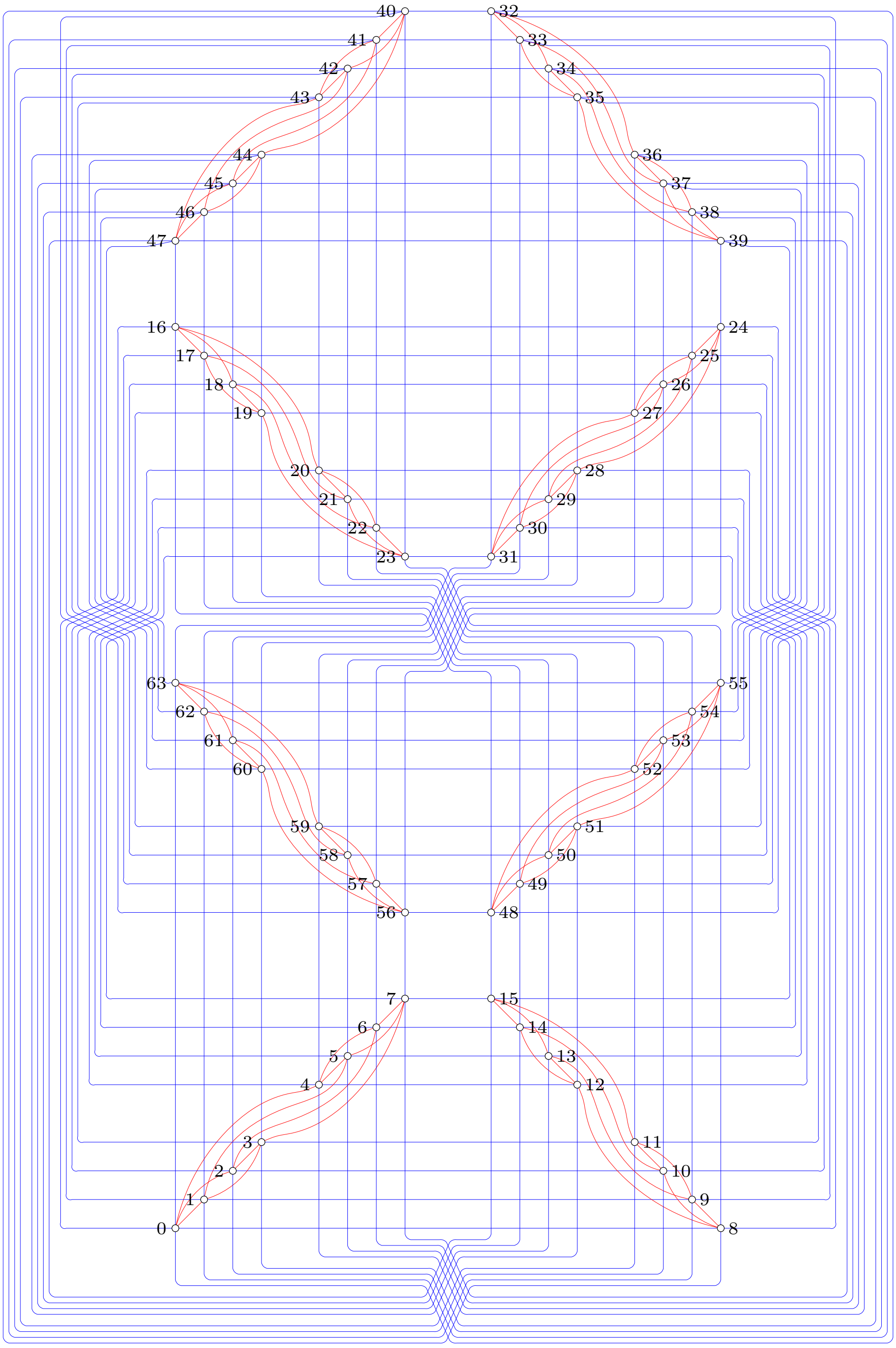

Figure 2.8: A drawing D_6 of FQ_6

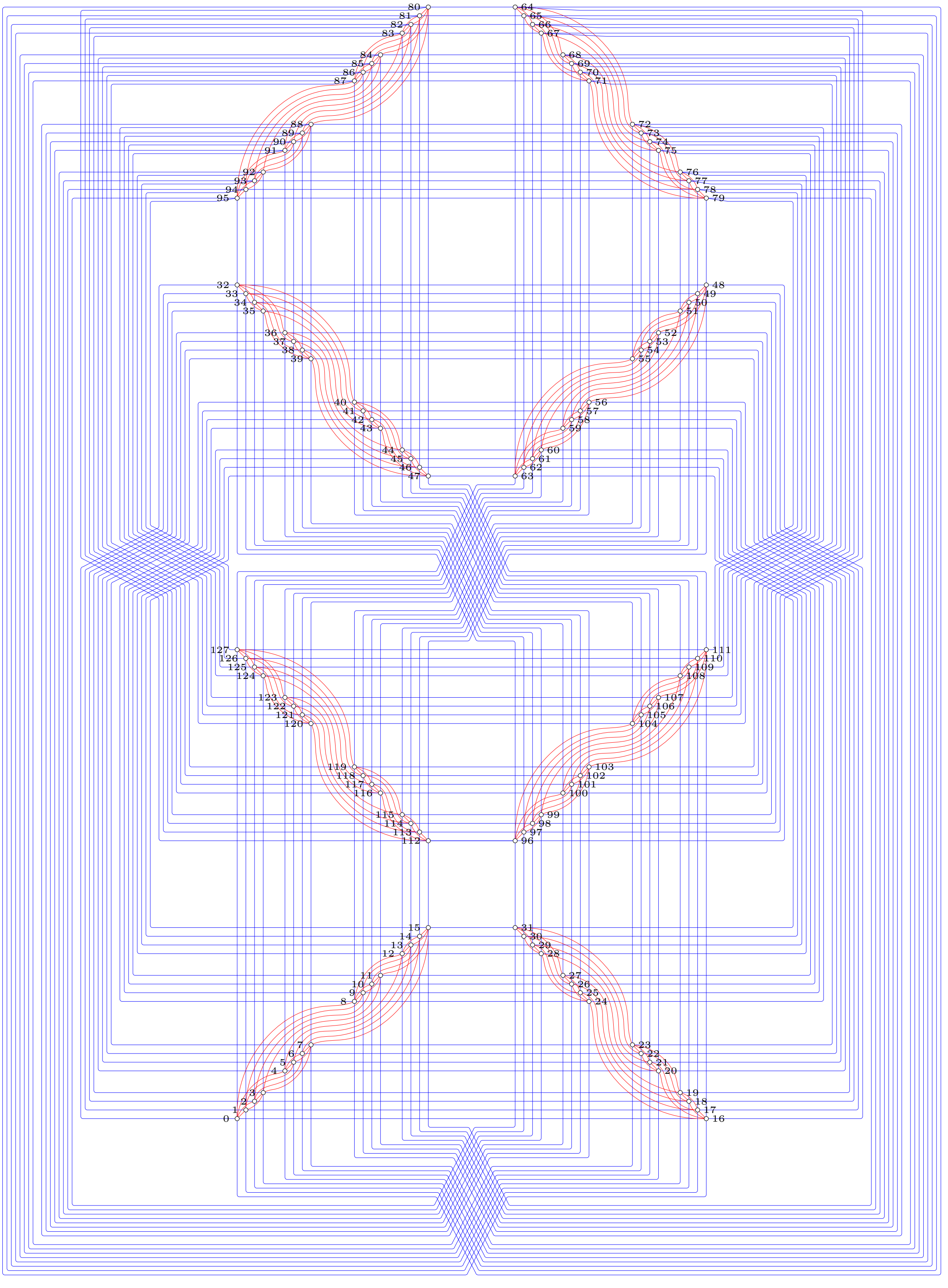

Figure 2.9: A drawing D_7 of FQ_7

3 Lower bound

We begin this section by giving some necessary definitions. Let $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_n$ be two vertices of FQ_n. Let

$$\theta_i(x) = x_i$$

for $i \in \{1, 2, \ldots, n\}$, and let

$$\mathcal{I}(x, y) = |\{i \in \{1, 2, \ldots, n\} : x_i = y_i\}|.$$

Moreover, suppose $xy \in E(FQ_n)$ is an edge, let $\mathrm{Dim}(xy) = 0$ if $y_1y_2\cdots y_n = \bar{x}_1\bar{x}_2\cdots\bar{x}_n$, and let $\mathrm{Dim}(xy)$ be the unique integer $t \in \{1, 2, \ldots, n\}$ such that $y_t = \bar{x}_t$ if otherwise.

The following definition is a key for the proof of the lower bound.

Definition 3.1. *For any two vertices $u, v \in V(FQ_n)$, let $\mathscr{P}_{u,v} = u_0u_1\ldots u_\ell$ be a path of FQ_n from u to v where $u_0 = u$ and $u_\ell = v$, such that if $\mathcal{I}(u, v) \leq \lfloor\frac{n}{2}\rfloor - 1$ then $\ell = \mathcal{I}(u, v) + 1$ and $0 = \mathrm{Dim}(u_0u_1) < \mathrm{Dim}(u_1u_2) < \cdots < \mathrm{Dim}(u_{\ell-1}u_\ell) \leq n$; and if $\mathcal{I}(u, v) \geq \lfloor\frac{n}{2}\rfloor$ then $\ell = n - \mathcal{I}(u, v)$ and $1 \leq \mathrm{Dim}(u_0u_1) < \mathrm{Dim}(u_1u_2) < \cdots < \mathrm{Dim}(u_{\ell-1}u_\ell) \leq n$.*

We shall introduce the lower bound method proposed by Leighton [9]. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. An embedding of G_1 in G_2 is a couple of mapping (φ, κ) satisfying

$$\varphi : V_1 \to V_2 \text{ is an injection}$$

$$\kappa : E_1 \to \{\text{set of all paths in } G_2\},$$

such that if $uv \in E_1$ then $\kappa(uv)$ is a path between $\varphi(u)$ and $\varphi(v)$. For any $e \in E_2$ define

$$cg_e(\varphi, \kappa) = |\{f \in E_1 : e \in \kappa(f)\}|$$

and

$$cg(\varphi, \kappa) = \max_{e \in E_2}\{cg_e(\varphi, \kappa)\}.$$

The value $cg(\varphi, \kappa)$ is called congestion.

Let $2K_m$ be the complete multigraph of m vertices, in which every two vertices are joined by two parallel edges.

Lemma 3.1. *[9] Let (φ, κ) be an embedding of G_1 in G_2 with congestion $cg(\varphi, \kappa)$. Let $\Delta(G_2)$ denote the maximal degree of G_2. Then*

$$cr(G_2) \geq \frac{cr(G_1)}{cg^2(\varphi, \kappa)} - \frac{|V_2|}{2}\Delta^2(G_2).$$

According to Erdős [5] and Kainen [8], the following lemmas hold.

Lemma 3.2. *[5] $cr(K_{2^n}) \geq \frac{2^n(2^n-1)(2^n-2)(2^n-3)}{80}$.*

Lemma 3.3. *[8] $cr(2K_{2^n}) = 4cr(K_{2^n})$.*

Now we are in a position to prove the lower bound of $cr(FQ_n)$.

Theorem 3.1.

$$cr(FQ_n) > \frac{4^n}{20 \times (1 - \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\lceil \frac{n}{2} \rceil + 1}})^2} - (n^2 + 2n + 4)2^{n-1}.$$

Proof. Let φ be an arbitrary bijection $V(2K_{2^n})$ onto $V(FQ_n)$. We define the mapping κ as follows. For any two vertices u and v of FQ_n, take $\mathscr{P}_{u,v}$ and $\mathscr{P}_{v,u}$ to be the images (paths) of the two parallel edges between $\varphi^{-1}(u)$ and $\varphi^{-1}(v)$ under κ.

Now we show that

$$cg_e(\varphi, \kappa) \leq 2^n - \binom{n}{\lfloor \frac{n}{2} \rfloor} \tag{1}$$

for each edge $e \in E(FQ_n)$ by two cases.

Let $e = xy$ be an arbitrary edge of FQ_n where $x = x_1 x_2 \cdots x_n$ and $y = y_1 y_2 \cdots y_n$.

Case 1. $\mathrm{Dim}(e) = 0$.

Consider the number of paths $\mathscr{P}_{u,v}$ traversing x previous to y. Since $\mathrm{Dim}(e) = 0$, we have $u = x$ and $\mathcal{I}(u, v) \leq \lfloor \frac{n}{2} \rfloor - 1$, which implies that there are exactly $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\lfloor \frac{n}{2} \rfloor - 1}$ choices of the ending vertex v, i.e., the number of paths traversing x previous to y is $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\lfloor \frac{n}{2} \rfloor - 1}$.

Similarly, the number of paths traversing y previous to x is $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\lfloor \frac{n}{2} \rfloor - 1}$.

Therefore, we conclude that $cg_e(\varphi, \kappa) = 2 \times (\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\lfloor \frac{n}{2} \rfloor - 1}) \leq 2^n - \binom{n}{\lfloor \frac{n}{2} \rfloor}$ for any edge e with $\mathrm{Dim}(e) = 0$.

Case 2. $\mathrm{Dim}(e) = t \in \{1, 2, \ldots, n\}$.

That is, $y_t = \bar{x}_t$ and $y_i = x_i$ for all $i \in \{1, 2, \ldots, n\} \setminus \{t\}$. Consider the number of paths $\mathscr{P}_{u,v}$ traversing x previous to y. Note that

$$\theta_i(v) = y_i \qquad \text{for all } i \in \{1, 2, \ldots, t\},$$

and that either

$$\text{(i)} \quad \theta_j(u) = x_j \qquad \text{for all } j \in \{t, t+1, \ldots, n\}$$

or

$$\text{(ii)} \quad \theta_j(u) = \bar{x}_j \qquad \text{for all } j \in \{t, t+1, \ldots, n\}.$$

If (i) holds, since $\mathcal{I}(u, v) \geq \lfloor \frac{n}{2} \rfloor$ and $\theta_t(u) \neq \theta_t(v)$, it follows that the number of choices of the pair of vertices (u, v) is

$$\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} \sum_{i=0}^{k} \binom{t-1}{i} \binom{n-t}{k-i} = \sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} \binom{n-1}{k}.$$

If (ii) holds, since $\mathcal{I}(u,v) \le \lfloor \frac{n}{2} \rfloor - 1$ and $\theta_t(u) = \theta_t(v)$, it follows that the number of choices of the pair of vertices (u,v) is

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 2} \sum_{i=0}^{k} \binom{t-1}{i}\binom{n-t}{k-i} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 2} \binom{n-1}{k}.$$

Hence, the number of paths traversing x previous to y is $\sum\limits_{k=0}^{\lfloor \frac{n}{2} \rfloor - 2} \binom{n-1}{k} + \sum\limits_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} \binom{n-1}{k} = 2^{n-1} - \binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1}$.

Similarly, the number of paths traversing y previous to x is $2^{n-1} - \binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1}$.

Therefore, we conclude that $cg_e(\varphi, \kappa) = 2 \times (2^{n-1} - \binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1}) \le 2^n - \binom{n}{\lfloor \frac{n}{2} \rfloor}$ for any edge e with $\mathrm{Dim}(e) \in \{1, 2, \ldots, n\}$.

From the above cases, we have (1) proved.

It is shown in [17] that $\binom{2k}{k} \ge \sqrt{\frac{2}{\pi}} \frac{2^{2k}}{\sqrt{2k+1}}$, and thus $\binom{2k-1}{k-1} = \frac{k}{2k}\binom{2k}{k} \ge \sqrt{\frac{2}{\pi}} \frac{2^{2k-1}}{\sqrt{2k+1}}$, where $k \in \mathbb{N}$. Therefore,

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \ge \sqrt{\frac{2}{\pi}} \frac{2^n}{\sqrt{2\lceil \frac{n}{2} \rceil + 1}} \tag{2}$$

for all $n \in \mathbb{N}$.

By (1), (2), Lemma 3.1, Lemma 3.2 and Lemma 3.3, we conclude that

$$\begin{aligned}
cr(FQ_n) &\ge \frac{2^n(2^n-1)(2^n-2)(2^n-3)}{20(2^n - \binom{n}{\lfloor \frac{n}{2} \rfloor})^2} - (n+1)^2 2^{n-1} \\
&\ge \frac{(2^n-1)(2^n-2)(2^n-3)}{20 \times 2^n (1 - \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\lceil \frac{n}{2} \rceil + 1}})^2} - (n^2 + 2n + 1)2^{n-1} \\
&= \frac{2^{3n} - 6 \times 2^{2n} + 11 \times 2^n - 6}{20 \times 2^n (1 - \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\lceil \frac{n}{2} \rceil + 1}})^2} - (n^2 + 2n + 1)2^{n-1} \\
&> \frac{4^n - 6 \times 2^n}{20 \times (1 - \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\lceil \frac{n}{2} \rceil + 1}})^2} - (n^2 + 2n + 1)2^{n-1} \\
&> \frac{4^n}{20 \times (1 - \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\lceil \frac{n}{2} \rceil + 1}})^2} - (n^2 + 2n + 4)2^{n-1}
\end{aligned}$$

□

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